# Longitudinal Injection Matching for the Low-Emittance Booster and a PhotoInjector Beam

—SRFEL-005—

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In SRFEL-003, I examined the evolution of a low-emittance, low-energy-spread, pico-second beam when injected into the booster. One consideration missing from the previous note is the proper matching of the longitudinal phase space. Because of the bunch parameters, a large rf voltage is needed. One result of this is that the matched longitudinal phase ellipse is tilted, so that the injected beam must be tilted in order to avoid phase-space dilution. In this note I'll derive the equations for the required tilt and show how to obtain the best match.

**TABLE 1. Some parameters of the low-emittance booster and photoinjector combination**

| | | | |
|---|---|---|---|
| Energy | 450 MeV | Voltage | <10MV |
| Frequency | 351.93 MHz | momentum compaction | 0.00144 |
| Harmonic number | 432 | Revolution time | 1.228 us |
| Equilibrium energy spread at 450MeV | 0.042% | Enery loss per turn at 450 MeV | 0.04 MeV |
| Injected bunch length | +/-0.75 ps | Injected energy spread | +/-0.02% |

Table 1 summarizes some relevant parameters of the low-emittance booster lattice and of the photoinjector beam proposed by Steve Milton. The first step is to compute the necessary rf voltage given the injected energy spread that allows maintaining the injected bunch length. Using the standard equations, one gets 25.6MV. This is beyond the capability of the booster rf system. Hence, we will apparently be unable to prevent dilution of the longitudinal emittance.

Because it might be useful should the above beam parameters change, it is worth showing how to match properly into such a high-voltage system. I start with the difference equations for an rf cavity followed by a single pass around a ring with momentum compaction $\alpha$ at energy E

$$\delta_{n+1} = \delta_n - \frac{V\cos\phi_s t_n \omega_{rf}}{E}$$

$$t_{n+1} = t_n + \alpha\delta_{n+1} T_o$$

where I'm using a linear approximation for the rf voltage as a function of time and assuming that the energy loss per turn is $V\sin\phi_s$.

To see why the matched ellipse is rotated, imagine that a beam with an unrotated phase ellipse enters the cavity and that the injected beam has bunch half-length length $\Delta t$ and half momentum spread $\Delta\delta$ as given in the table above. If this beam is matched, then any particle chosen on the ellipse $\left(\frac{t}{\Delta t}\right)^2 + \left(\frac{\delta}{\Delta\delta}\right)^2 = 1$ will still be on that ellipse after 1 pass. Taking $t = \Delta t$ and $\delta = 0$, I get after a single turn 0.87 instead of 1 for the "invariant". This is saying that the matched ellipse equation needs a term proportional to $\delta t$.

To derive the matched ellipse, I start with the distribution function for the beam, given by $\psi = Ne^{-J/2}$, where $J = a\delta^2 + 2b\delta t + ct^2$. The usual RMS parameters of the beam are

$$\langle\delta^2\rangle = \frac{c}{ac-b^2} \qquad \langle t^2\rangle = \frac{a}{ac-b^2} \qquad \langle t\delta\rangle = \frac{-b}{ac-b^2}$$

I want to find a, b, and c such that the ellipse is invariant under the transformation given above. Assuming for convenience that the value of $\sigma_t$ is fixed, one obtains

$$c = \frac{K}{\sigma_t^2} \qquad a = \frac{\alpha E T_o}{V\cos\phi_s\omega_{rf}}c \qquad b = \frac{-\alpha T_o}{2}c$$

$$K = \frac{1}{1 - \frac{\alpha V\cos\phi_s\omega_{rf}T_o}{4E}}$$

Using these results, one can express the RMS properties of the matched beam as

$$\langle\delta^2\rangle = \frac{V\cos\phi_s\omega_{rf}\sigma_t^2}{\alpha E T_o} \qquad \langle t\delta\rangle = \frac{1}{2}\alpha T_o\langle\delta^2\rangle$$

In the limit of $\frac{V}{E} \to 0$, $\frac{\langle t\delta\rangle}{\sqrt{\langle\delta^2\rangle\langle t^2\rangle}} \to 0$, so that the ellipse is untilted as we expect. In passing, I note that when tracking with the program **elegant**, a parameter **dp_s_coupling** may be given to set up the tilt of the longitudinal ellipse. This parameter is defined as

$$\frac{\langle t\delta\rangle}{\sqrt{\langle t^2\rangle\langle\delta^2\rangle}} = \frac{1}{2}\sqrt{\frac{\alpha T_o V\cos\phi_s\omega_{rf}}{E}}$$

These equations tell us what voltage, energy spread, and time-energy correlation to choose for a given bunch length. Assume that one starts with an uncorrelated distribution with given $\sigma_t$ and $\sigma_\delta$, and that one rotates this distribution (using null phasing in the linac, for example) according to $\delta \to \delta + Rt$. The new beam parameters are easily derived, and using the above results one can solve for the rotation factor R and the voltage:

$$R = \frac{1 - \sqrt{1 - \left(\frac{\alpha\sigma_\delta T_o}{\sigma_t}\right)^2}}{\alpha T_o}$$

$$V = \frac{2E}{\cos\phi_s \omega_{rf}} R$$

Plugging in values from the above table (using $\Delta t$ for $\sigma_t$ etc.) gives $R = 6.68\times10^7 s^{-1}$ and $V = 27.2MV$. Not surprisingly, this is comparable to the voltage requirement computed before using standard formulae. Note that the voltage $V_R = \frac{RE}{2\pi f_{inj}}$ required to produce the time-energy correlation using a $f_{inj} = 2856$ MHz cavity is about 1.7MV, which is easily achieved. The tilt of the matched ellipse can be evaluated by looking at $\frac{\langle t\delta\rangle}{\sqrt{\langle t^2\rangle}}$, which has value 0.005%, compared to an energy spread of +/- 0.02%.

Given that a true match is impossible, the next question is how to minimize the mismatch. We may assume that if a beam is mismatched, it will eventually filament and thus increase in longitudinal emittance.

One approach is to change the bunch length so that R corresponds to a value consistent with 10MV. This requires $R = 2.46\times10^7 s^{-1}$ and $\Delta t = 1.2ps$, a 60% increase in bunch length at thus a 60% drop in peak current. An alternative would be to drop the energy spread to 0.012%, but this may be unachievable. The required rotation can be achieved easily with 0.6MV in a 2856 MHz cavity.

Another approach is to find the voltage that minimizes the phase-space dilution for the given beam parameters. It turns out, not surprisingly, that the answer is to use the maximum rf voltage. However, it is worth writing down the results for use in the event that the beam parameters are changed. For this computation, I refer to the figure below. The phase ellipse with width $\Delta t_o$ and height $\Delta\delta_o$ rotates into an ellipse with width

$\Delta t_1 = \Delta\delta_o\sqrt{\dfrac{\alpha E T_o}{V\cos\phi_s\omega_{rf}}}$ and height $\Delta\delta_1 = \Delta t_o\sqrt{\dfrac{V\cos\phi_s\omega_{rf}}{\alpha E T_o}}$. Because of dilution, the eventual phase space occupied is

$$Max(\Delta\delta_o\Delta t_1, \Delta\delta_1\Delta t_o) = Max\left(\Delta\delta_o^2\sqrt{\frac{\alpha E T_o}{V\cos\phi_s\omega_{rf}}}, \Delta t_o^2\sqrt{\frac{V\cos\phi_s\omega_{rf}}{\alpha E T_o}}\right)$$

The goal is to minimize this function by adjusting V. Plugging in numbers from the table, one can express this function as

$$Max\left(\frac{7.59\times10^{-16}}{\sqrt{V}}, 2.97\times10^{-17}\sqrt{V}\right)$$

where V is in MV. The cross over of the two terms is at 25.5MV, below which the first term is greater. Since we can't get above 10MV, we should minimize the first term, which requires maximizing V, as expected. In other words, we are minimizing the extent to which the energy spread turns into bunch length.